INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 42, Northern Spring 2021 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The number $2021 = 43 \cdot 47$ is composite. Prove that if we insert any number of digits "8" between 20 and 21 then the number remains composite. (4 points)

2. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile every child divides the current number of sweets in the pile by the number of children currently in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children approach the pile. (5 points)

3. There is an equilateral triangle ABC. Let E, F and K be points such that E lies on side AB, F lies on side AC, K lies on the extension of side AB and $AE = CF = BK$. Let P be the midpoint of segment EF. Prove that angle KPC is a right angle. (6 points)

4. A traveller arrived at an island inhabited by 50 natives. All the natives stood in a circle and each announced first the age of his left neighbour, then the age of his right neighbour. Each native is either a knight who gave both numbers correctly or a knave who increased one of the numbers by 1 and decreased the other by 1 (by his or her choice). Is it always possible, after such an introduction, for the traveller to determine who of the natives are knights and who are knaves? (7 points)

5. In the center of each square cell of a checkered rectangle M there is a point-sized light bulb. All the bulbs are initially switched off. In one turn we are allowed to choose a straight line not intersecting any light bulb such that on one side of it all the bulbs are switched off, and to switch all of them on. In each turn at least one bulb should be switched on. The task is to switch on all the light bulbs using the largest possible number of turns. What is the maximum number of turns if:

(a) M is a square of size 21×21 cells; (4 points)

(b) M is a rectangle of size 20×21 cells? (4 points)

6. 100 tourists arrive at a hotel at night. They know that in the hotel there are single rooms numbered $1, 2, \ldots, n$, and among them k rooms (the tourists do not know their numbers) are under repair, the other rooms are vacant. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. Tourists do not see which rooms are occupied by others but they do not want to bother their peers and check already occupied rooms. Therefore they coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may occupy their rooms for sure. (10 points)

7. Let p and q be two coprime positive integers. A frog hops along the integer line starting from 0. On every hop it moves either p units to the right or q units to the left. Eventually, the frog returns to the initial point. Prove that for every positive integer d with $d < p + q$ there exist two numbers visited by the frog which differ by d units. (12 points)